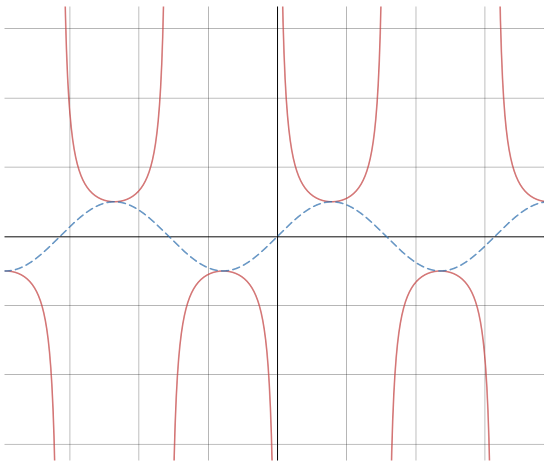
Quiz on Chapter 3 - Solutions

MATH 1110

2016.10.26

Problems

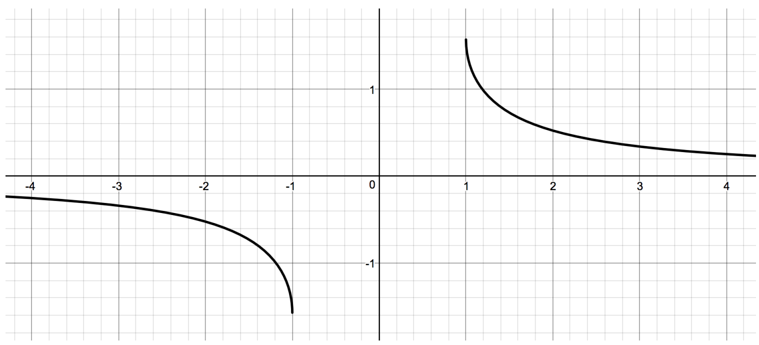
Problem 1. Find the formula for the derivative of $\csc^{-1}(x)$. (**Warning:** the restricted domain will consist of two intervals.)



Solution 1 We first restrict to a domain D on which $\csc(x)$ is injective and the derivative of $\csc(x)$ is non-zero.

$$D = (-\pi/2, 0) \sqcup (0, \pi/2)$$

We have the following sketch of the graph of $\csc^{-1}(x)$ corresponding to this choice of domain.



Since the derivative of $\csc(x)$ is not zero on our domain we can apply the formula for the derivative of the inverse function.

$$\frac{d}{dx}(\csc^{-1}(x)) = \frac{1}{\csc'(\csc^{-1}(x))} \qquad (\text{where } \csc'(x) \neq 0)$$

We have that $\csc'(x) = -\csc(x)\cot(x)$. Recall the trig identity (obtained from $\sin^2(x) + \cos^2(x) = 1$ by dividing throughout by $\sin^2(x)$)

$$1 + \cot^2(x) = \csc^2(x)$$

Putting this together gives

$$\begin{aligned}\frac{d}{dx}(\csc^{-1}(x)) &= -\frac{1}{\csc(\csc^{-1}(x))\cot(\csc^{-1}(x))}\\ &= -\frac{1}{\pm x\sqrt{\csc^2(\csc^{-1}(x))-1}}\\ &= -\frac{1}{\pm x\sqrt{x^2-1}}\end{aligned}$$

It remains to choose the appropriate signs in the expression for the derivative. By inspection of the graph of $y = \csc^{-1}(x)$ we see that the derivative is always negative, and thus,

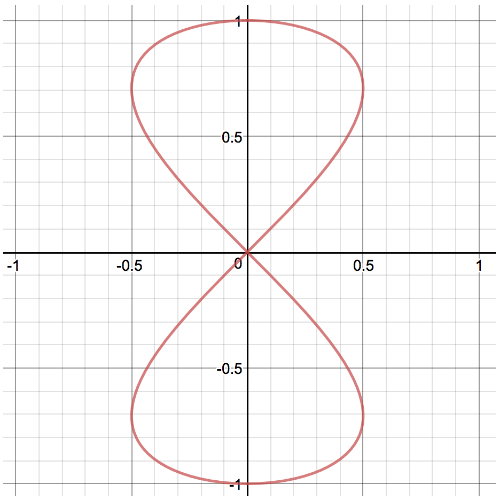
$$\frac{d}{dx}(\csc^{-1}(x)) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Problem 2. Find the equation for the slope of the tangent line to the curve $y^4 = y^2 - x^2$ at a point (x_0, y_0). What are the points where the slope of the tangent line is 0?

Solution 2 We differentiate implicitly to obtain,

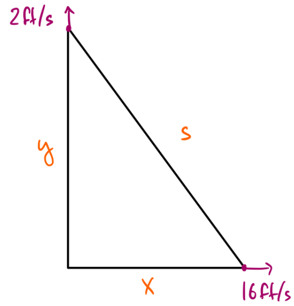
$$4y^3 \cdot y' = 2y \cdot y' - 2x \qquad \Rightarrow \qquad y' = \frac{-2x}{4y^3 - y}$$

We see that the slope is zero if and only if $x = 0$ (and $y \neq 0$). It remains to find all the points on the curve with $x = 0$. That is we need to solve $y^4 = y^2$ from which we see that $y = \pm 1$. Thus the two points $(0, 1)$ and $(0, -1)$ are the only points on the curve with slope equal to 0. Here is a sketch of the curve in the question.



Problem 3. Harry Potter casts a spell to make his aunt Marjorie fill in with air and fly up to the sky at a constant speed 2 feet/sec. At first, everyone just stares at aunt Marjorie rising: members of Dursley family are in shock, Harry is silently happy. As soon as aunt Marge reaches 50 feet above the ground, Harry Potter feels he should run away, so he starts running at a constant speed of 16 ft/sec. How fast is the distance between Harry and aunt Marge growing exactly 5 seconds after Harry started to run?

Solution 3 This somewhat comical situation is nicely represented by a right angle triangle whose hypotenuse, $s = s(t)$, is the distance between Harry and his aunt. Ley $y = y(t)$ be the height of aunt Marge and $x = x(t)$ the position of Harry at time t.



Let $t = 0$ be the moment that Harry decides to run. So $y(0) = 50\text{ft}$ and $x(0) = 0$. We are interested in the moment $t = 5$ where we have that $y(5) = 50 + 5 \cdot 2 = 60\text{ft}$ and $x(5) = 5 \cdot 16 = 80\text{ft}$ and $s(5) = \sqrt{60^2 + 80^2} = 100$. We are tasked to find $s'(t)$. To this end consider the relationship imposed on y, x and s by Pythagoras' theorem,

$$s^2 = y^2 + x^2$$
$$2s \cdot s' = 2y \cdot y' + 2x \cdot x'$$

At $t = 5$ we have

$$2(100)s'(5) = 2(60) \cdot 2 + 2(80) \cdot 16 \qquad \Rightarrow \qquad s'(5) = \frac{2800}{200} = 14\text{ft/s}$$

Problem 4. Pinocchio is running for President of his High school class. He is giving a speech promising that, if elected, he will make the minimum grade a student can get to be A-, free snacks at the school diner, and many other nice things. Lies constitute around 50% of the total number of words Pinocchio is saying. With every word of a lie, Pinocchio's nose (which is cylindrical of initial diameter 1 inch and length 3 inches) is growing in length by 0.5 inches and by 0.05 inches in diameter. How fast is the volume of his nose growing after 10 minutes of his speech if you know that Pinocchio says about 150 words per minute.

Solution 4 Pinocchio's nose is of length $l = l(t)$ and diameter $d = d(t)$. We are interested in the volume of the nose after 10 minutes of the speech. At $t = 10$ we can calculate the diameter and the length of his nose as follows. We'll assume Pinocchio says 1500 words in 10 minutes, 750 of which are lies. We thus have that the length of the nose increases by $750 \cdot 0.5 = 375\text{in}$ to give $l(10) = 753\text{in}$ and the diameter increases by $750 \cdot 0.05 = 75/2\text{in}$ to give $d(10) = 77/2\text{in}$. The volume of the nose is related to the length and diameter via the following relationship.

$$V = \pi(d/2)^2 l = \frac{\pi}{4} d^2 l$$

Taking the derivative using the chain rule,

$$V' = \frac{\pi}{4}(2d \cdot d' \cdot l + d^2 \cdot l')$$

We have all the information to evaluate at $t = 10$:

$$V'(10) = \frac{\pi}{4}(2(77/2)(0.05)(735) + (77/2)^2(0.5)) = \text{really?}$$